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On expandable discrete collections

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Abstract

Expandability-type properties, which are more general than both normality and countable paracompactness, are considered. We give a common generalization of theorems related to closed mappings of normal or countably paracompact spaces onto a q-space. We also consider these properties in product spaces and prove theorems which are parallel to Katětov's theorem on hereditary normality in product spaces. An application to a space of closed subsets is given too.

Keywords: Normality; Countable paracompactness; Closed mapping; q-space; ω-expandability; Expandability-type properties; Space of closed subsets

AMS classification: 54B10; 54C10; 54D20

1. Introduction

Let f be a closed mapping of a space X onto a q-space Y. Then the boundary of $f^{-1}(y)$ is countably compact for each $y \in Y$ if X is normal [17] or countably paracompact T_1-space [12]. We consider a generalization of normality and countable paracompactness which permits us to give a common extension of these two theorems (Theorem 2.1 below). Another group of results are connected with Katětov's famous theorem [8, 2.7.15(a)] on hereditary normality in product spaces. We prove analogous theorems for expandability-type properties (Section 3). We consider also infinite products and generalize the next theorem of Efimov [8, 3.12.12(k)]: every hereditarily normal dyadic space is metrizable. In Section 5 we present results connected with the theorem from [2]: if X is compact and $X^2 \setminus \Delta$ is normal then X is first-countable. We note here that the last result was inspired by Gruenhage's theorem [9] that paracompactness of $X^2 \setminus \Delta$ implies metrizability of X for compact X. The last Section 6 contains an application of preceding results to a space of closed subsets.

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All spaces are assumed to be T_1, all mappings are continuous. An ordinal is the set of smaller ordinals and cardinal is an initial ordinal. Another terminology and notations not defined in this paper can be found in [8].

A topological space X is said to be ω-expandable if every locally finite collection $\{F_i\colon\ i<\omega\}$ of subsets of X is expandable, i.e., there exists a locally finite collection $\{G_i\colon\ i<\omega\}$ of open subsets of X such that $F_i \subseteq G_i$ for every $i<\omega$ [22]. Of course G_i and G_j are considered as different sets if $i \neq j$. Mansfield showed that a Hausdorff space X is countably paracompact if and only if X is ω-expandable [8, 5.5.17].

Definition 1.1. A space X is said to have property sE (respectively E) if every discrete collection $\{F_i\colon\ i<\omega\}$ of closed countable subsets of X (respectively of singletons) is expandable.

Recall that a space is said to be pseudonormal if every countable closed subset has arbitrarily small closed neighborhoods [20]. A space is called quasinormal [1] or to have property D [7] if every countable closed discrete set has arbitrarily small closed neighborhoods. A regular space is said to have property wD [7] if every infinite closed discrete set has an infinite subset which has arbitrarily small closed neighborhoods.

Definition 1.2. A space X is said to have property wE if every infinite closed discrete subset of X has an infinite subset which is expandable.

Clearly,

$$\begin{array}{lclcl} \text{normal} & \Rightarrow & \text{pseudonormal} \Rightarrow & D \Rightarrow & wD \\ & & \Downarrow & \Downarrow & \Downarrow \\ \text{countably paracompact} & \Rightarrow & sE \qquad \Rightarrow & E \Rightarrow & wE. \end{array}$$

Every regular space with property E (respectively wE) possesses D (respectively wD) [7, Proposition 12.1]. The well-known Tychonoff Plank

$$((\omega_1+1)\times(\omega+1))\setminus\{(\omega_1,\omega)\}$$

is an example of a completely regular space without wE. The space

$$(\omega_1\times(\omega+1))\setminus(\mathrm{LIM}\times\{\omega\}),$$

where $\mathrm{LIM}=\{\alpha<\omega_1\colon\ \alpha \text{ is limit}\}$, is completely regular and satisfies all properties from the diagram above except normality and countable paracompactness.

We note here that in the class of completely regular spaces the property E coincides with the property of well-separatedness, introduced by Hansard [11], and the property wE coincides with the notion of ss-discreteness, introduced by Isiwata [13]. But there exist nonregular (and hence non-D and non-wD) countably paracompact (and hence with properties E and wE) spaces [8].

Let Q be a topological property (a class of topological spaces). A space X is said to be point-Q (respectively F_σ-Q; ∇-Q), if for every $x\in X$ a subspace $X\setminus\{x\}$ (respectively every F_σ-subset of X; the subset $X^2 \setminus \Delta$, where $\Delta = \{(x,x)\colon x \in X\}$ is a diagonal) has the property Q.

2. Closed mappings onto q-spaces

Recall that $y \in Y$ is a q-point if y has a sequence $\{N_i\colon i < \omega\}$ of open neighborhoods such that whenever $y_i \in N_i$ and $y_i \neq y_j$ if $i \neq j$, the set $\{y_i\colon i < \omega\}$ has a limit point in Y. A space Y is a q-space [17], if each of its points is a q-point. The next theorem is a common generalization of Corollary 2.2 from [17] and Theorem 1.1 from [12].

Theorem 2.1. *Let X be wE, Y a q-space and $f\colon X \to Y$ a closed mapping onto. Then* $\operatorname{Fr} f^{-1}(y)$ *is countably compact for each $y \in Y$.*

Proof. Suppose that $\operatorname{Fr} f^{-1}(y)$ is not countably compact. Choose a closed discrete infinite subset $D \subseteq \operatorname{Fr} f^{-1}(y)$. Let us take an infinite subset $\{x_i \in D\colon i < \omega\}$ and a locally finite collection $\{G_i\colon i < \omega\}$ of open subsets of X such that $x_i \in G_i$ for every $i < \omega$. Fix neighborhoods N_i, $i < \omega$, from the definition of a q-point y. This point y is not isolated, because $\operatorname{Fr} f^{-1}(y) \neq \emptyset$. Then all subsets N_i are infinite. Let us take $z_1 \in (G_1 \cap f^{-1}(N_1 \setminus \{y\}))$. Then $f(z_1) = y_1 \in N_1$ and $y_1 \neq y$. Suppose that points z_i, $i \leqslant n$, are defined in such manner that $f(z_i) = y_i \in N_i$, $y_i \neq y$, $i \leqslant n$, and $y_i \neq y_j$ if $i \neq j$. Let choose a point

$$z_{n+1} \in G_{n+1} \cap f^{-1}\bigl(N_{n+1} \setminus \{y,\ y_1, y_2, \dots, y_n\}\bigr)$$

and let $y_{n+1} = f(z_{n+1})$. It is clear that $y_{n+1} \in N_{n+1}$, $y_{n+1} \neq y$ and $y_{n+1} \neq y_j$ if $j \leqslant n$. Since the collection $\{G_i\colon i < \omega\}$ is locally finite the set $Z = \{z_i\colon i < \omega\}$ is a closed discrete subset of X. A set Z is closed and discrete if and only if every subset of Z is closed. Since f is closed every subset of $f(Z)$ is closed too, so $f(Z)$ is discrete in Y. But this is contradiction with the definition of a q-point $y \in Y$. $\square$

Remark 2.2. The property wE in the theorem 2.1 for a space X is essential as the next simple example shows. Let X be the Tychonoff Plank $((\omega_1+1)\times(\omega+1))\setminus\{(\omega_1,\omega)\}$ and $N = \{(\omega_1, n)\colon n < \omega\}$. Let Y be the quotient space obtained from X by identifying the set N to a point and let $f\colon X \to Y$ be the natural mapping. As we know the mapping f is closed. The space Y is a q-space. Clearly, $\operatorname{Fr} N = N$. But N is not countably compact, and the space X has not the property wE. Nevertheless we note here that by Michael's Theorem [17, Theorem 2.1] every continuous real-valued function on X is bounded on N.

3. Around Katětov's theorem

According to Katětov's well-known theorem [8, 2.7.15(a)] if $X \times Y$ is hereditarily normal, and Y contains a countable nonclosed subset, then every closed subset of

X is a G_δ-set. Of course we can not replace hereditary normality even by hereditary pseudonormality, because the product $\omega_1 \times (\omega+1)$ is hereditarily pseudonormal (and hence hereditarily sE), but LIM is not a G_δ-set in ω_1. Nevertheless we see that every countable closed subset of ω_1 is a G_δ-set and we have more general

Theorem 3.1. *If $X \times Y$ is hereditarily sE, Y is regular and contains a countable nonclosed subset, then every countable closed subset of X is a G_δ-set.*

This theorem is an immediate consequence of the next Lemma 3.2.

Lemma 3.2. *Let $\{y\} \cup M$ be a countable regular space with a nonisolated point y, a set H be a countable closed subset of X, and $Z = (X \times M) \cup ((X \setminus H) \times \{y\})$ is sE. Then H is a G_δ-set in X.*

Proof. There exist open-closed in $M_0 = \{y\} \cup M$ neighborhoods M_i of point y such that $\{y\} = \bigcap\{M_i\colon i < \omega\}$ and $M_{i+1} \subset M_i$ for every $i < \omega$ [8, 6.2.8]. Then $F_i = H \times (M_{i-1} \setminus M_i)$, $i < \omega$, are countable closed subsets of Z. Let us show that the collection $\{F_i\colon i < \omega\}$ is discrete in Z. Let $(x, m) \in Z$. If $x \notin H$, then the open set $(X \setminus H) \times M_0$ is neighborhood of (x, m), which does not intersect F_i for every $i < \omega$. If $x \in H$, then $m \neq y$. Let $i(m) = \min\{i\colon m \notin M_i\}$. Then $X \times (M_{i(m)-1} \setminus M_{i(m)})$ is a neighborhood of (x, m), which intersects only $F_{i(m)}$. There exists a locally finite collection $\{G_i\colon i < \omega\}$ of open subsets of Z such that $F_i \subseteq G_i$ for every $i < \omega$. Let $m \in M$. The set $W_m = \{x \in X\colon (x, m) \in G_{i(m)}\}$ is open in X. If $x \in H$, then $(x, m) \in F_{i(m)} \subset G_{i(m)}$, because $m \in M_{i(m)-1} \setminus M_{i(m)}$. Clearly, $x \in W_m$ for every $m \in M$. Thus $H \subseteq \bigcap\{W_m\colon m \in M\}$. If $x \notin H$, then we consider the point (x, y). Let $U \times V$ be a neighborhood of (x, y), which intersects only a finite number of sets G_i. Let $N = \max\{i\colon (U \times V) \cap G_i \neq \emptyset\}$ and let $m \in M_{N+1} \cap V$. Then $(U \times V) \cap G_{i(m)} = \emptyset$, because $i(m) \geqslant N+1$. But $(x, m) \in U \times V$. Thus $(x, m) \notin G_{i(m)}$ and $x \notin W_m$. So $H = \bigcap\{W_m\colon m \in M\}$. $\square$

Remark 3.3. It is obvious that

Hereditarily $sE \Rightarrow$ point-F_σ-$sE \Rightarrow$ point-F_σ-$E \Rightarrow$ point-$E \Rightarrow$ point-wE.

Example 3.4. Let R_Q be the well-known Michael's example [8, 5.1.32], i.e., R_Q be the real line R with topology generated by the base $\{U\colon U$ is open in R or U is a subset of $R \setminus Q\}$, where Q is the subspace consisting of all rational numbers. Then the product $R_Q \times (\omega+1)$ is point-paracompact and hence point-F_σ-sE, but the countable closed set Q is not G_δ in R_Q. Thus $R_Q \times (\omega+1)$ is not hereditarily sE by Theorem 3.1. We note also that every point of this product is a G_δ-point.

Theorem 3.5. *If $X \times Y$ is point-F_σ-sE, a space Y is regular and contains a countable nonclosed subset, then every point of X is a G_δ-point.*

Proof. Let us take $H = \{x\}$ in the Lemma 3.2. Then Z is a F_σ-set in $(X \times Y) \setminus \{(x, y)\}$ and hence is sE. Consequently x is a G_δ-point in X. $\square$

Lemma 3.6. *Let $\{y\} \cup M$ be a countable space with a single nonisolated point y, and $x \in X$. Let $Z = (X \times M) \cup ((X \setminus \{x\}) \times \{y\})$ be E. Then x is a G_δ-point in X.*

Proof. The set $\{(x,m)\colon m \in M\}$ is discrete in Z, then there exists a locally finite collection $\{G_m\colon m \in M\}$ of open subsets of Z such that $(x,m) \in G_m$ for every $m \in M$. Define the set $W_m = \{p \in X\colon (p,m) \in G_m\}$. The set W_m is open in X and $x \in W_m$ for every $m \in M$. If $p \neq x$ then choose the neighborhood $U \times V$ of the point (p,y) such that the set $F = \{m \in M\colon (U \times V) \cap G_m \neq \emptyset\}$ is finite. Let us take $m \in (M \setminus F) \cap V$. It is not difficult to see that $p \notin W_m$. So $\{x\} = \bigcap\{W_m\colon m \in M\}$. □

The next theorem is a consequence of Lemma 3.6.

Theorem 3.7. *If $X \times Y$ is point-F_σ-E and Y contains a countable nonclosed discrete subset, then every point of X is a G_δ-point.*

A compact space X is first-countable if and only if X^2 is point-normal [2]. Every normal space is F_σ-normal [8, 2.1.E], so the property point-F_σ-E is weaker than point-normality and we have the next more general assertion as a corollary of Theorem 3.7.

Corollary 3.8. *A compact space X is first-countable if and only if X^2 is point-F_σ-E.*

Proof. Note that every infinite compact space contains a countable nonclosed discrete subset. □

Theorem 3.9. *If $X \times Y$ is point-E and Y contains a countable nonclosed subset M with a single limit point $y \in Y$, then every point of X is a G_δ-point.*

Proof. The set $Z = (X \times M) \cup ((X \setminus \{x\}) \times \{y\})$ is a closed subset of $(X \times Y) \setminus \{(x,y)\}$. Then Z is E, because E is hereditary with respect to closed subsets. Now apply 3.6. □

The proof of the next Lemma 3.10 is analogous to the proof of the Lemma 3.6.

Lemma 3.10. *Let $Z = (X \times \omega) \cup ((X \setminus \{x\}) \times \{\omega\})$ be a subspace of the product $X \times (\omega + 1)$. If Z is wE, then x is a G_δ-point in X.*

The next theorem is a consequence of Lemma 3.10.

Theorem 3.11. *If $X \times Y$ is point-wE and Y contains $\omega + 1$, then every point of X is a G_δ-point.*

Corollary 3.12. *Let X be an infinite compact space. Then X is first-countable if only if $X \supseteq \omega + 1$ and X^2 is point-wE.*

Remark 3.13. The condition $X \supseteq \omega + 1$ in the above corollary is essential, because $(\beta\omega \setminus \omega)^2$ is point-(countably compact) and hence point-E, but $\beta\omega \setminus \omega$ is not first-countable.

4. Infinite products and dyadic spaces

We shall now prove that no uncountable products of spaces having at least two points are hereditarily wE. Let $X = \prod\{X_\xi\colon \xi \in \Xi\}$, where every X_ξ contains more than one point. Let $x^* = \{x^*_\xi\} \in X$ and $Q(y) = \{\xi \in \Xi\colon y_\xi \neq x^*_\xi\}$ for every point $y = \{y_\xi\}$ from X. Let τ be an infinite cardinal number. Recall that the subspace $Y(x^*, \tau) = \{y \in X\colon |Q(y)| < \tau\}$ is called a τ-envelope of the spaces X_ξ, $\xi \in \Xi$. Of course, $Y(x^*, \tau) = X$, if $|\Xi| < \tau$. In the case $\tau = \omega_1$ (respectively $\tau = \omega$) we have a Σ-product (respectively σ-product) of the spaces X_ξ. The next proposition is an easy consequence of the definition of Tychonoff topology on X.

Proposition 4.1. *If Y is a τ-envelope of spaces X_ξ, $\xi \in \Xi$, having at least two points, $|\Xi| \geqslant \omega_1$, and $z \in Y$, then z is not a G_δ-point in Y.*

Proposition 4.2. *If Y is a τ-envelope of spaces X_ξ, $\xi \in \Xi$, having at least two points, $|\Xi| \geqslant \max\{\omega_1, \tau\}$, then a subspace $Y \setminus \{z\}$ is not wE for every $z \in Y$.*

Proof. For every $P \subseteq \Xi$, put $Y_P = \{y \in Y\colon y_\xi = x^*_\xi \text{ for } \xi \in \Xi \setminus P\}$. Let us take $P \subseteq \Xi$ such that $Q(z) \subset P$ and $|P| = |\Xi \setminus P| = |\Xi|$. It is possible, because $|\Xi| \geqslant \tau$. It is easy to see that the space Y is homeomorphic to the product $Y_P \times Y_{\Xi\setminus P}$. The space $Y_{\Xi\setminus P}$ contains $\omega + 1$, because $|\Xi \setminus P| \geqslant \omega_1$. Of course we can assume that $z = (p, \omega)$, where $p \in Y_P$ and $\omega \in \omega + 1 \subset Y_{\Xi\setminus P}$. The subspace $(Y_P \times \omega) \cup ((Y_P \setminus \{p\}) \times \{\omega\})$ is closed in $Y \setminus \{z\}$. If the last space is wE, then p is a G_δ-point in Y_P by Lemma 3.10. But this is a contradiction with Proposition 4.1. $\square$

As corollaries of Proposition 4.2 we have nonnormality of $\Sigma \setminus \{x\}$, where Σ is a Σ-product of uncountably many spaces each having more than one point and $x \in \Sigma$ [5], and nonnormality of $\sigma \setminus \{x\}$, where σ is a corresponding σ-product and $x \in \sigma$ [3,14].

Remark 4.3. In [14] author proved that if a set Z is closed in the τ-envelope $Y(x^*, \tau)$ and $|\bigcup\{Q(z)\colon z \in Z\}| < \tau$, then $Y \setminus Z$ is a nonnormal subset of Y. The author does not know if $Y \setminus Z$ is non-wE in this case.

The next theorem is a slight generalization of the theorem of Efimov [8, 3.12.12(k)], who proved that every hereditarily normal dyadic space is metrizable.

Theorem 4.4. *Every point-wE dyadic space is metrizable.*

Proof. If X is a dyadic space, then there exists a mapping f of Cantor cube $D^{w(X)}$ onto X. Let Σ be a Σ-product, lying in $D^{w(X)}$. If $w(X) \geqslant \omega_1$, then there exists a point $x \in X \setminus f(\Sigma)$ [8, 3.12.23(f)]. The Σ-product Σ is countably compact and hence pseudocompact. So does $f(\Sigma)$. But $f(\Sigma)$ is dense in $X \setminus \{x\}$. Hence $X \setminus \{x\}$ is pseudocompact and is not countably compact [8, 3.12.12(i)]. But every completely regular pseudocompact wE space is countably compact (see 2.5 from [13]). This contradiction shows that $w(X) < \omega_1$. $\square$

5. ∇-properties

In [2] it is proved that every ∇-normal compact space is first-countable and hence every ∇-normal dyadic space is metrizable. Every dyadic space is a $\varkappa$-space [8, 3.12.12(i)]. This fact gives us a simple proof of first-countability (and metrizability) of ∇-wE dyadic space. Recall that a point $x \in X$ is called a $\varkappa$-point if it is a limit of a convergent sequence of points $x_n \in X \setminus \{x\}$ [1]. A space X is called a $\varkappa$-space if every nonisolated point $x \in X$ is a $\varkappa$-point.

Theorem 5.1. *If a compact $\varkappa$-space X is ∇-wE, then X is first-countable.*

This theorem is a straightforward consequence of the next Lemma 5.2, which is similar to Lemma 2 from [10].

Lemma 5.2. *Let X be ∇-wE, $x \in X$ and $x = \lim\{x_n \in X \setminus \{x\}\colon n < \omega\}$. Then x is a G_δ-point of X.*

Proof. The set $\{(x_n, x)\colon n < \omega\}$ is closed and discrete in $X^2 \setminus \Delta$. So there exist an infinite set $E \subseteq \omega$ and a locally finite in $X^2 \setminus \Delta$ system of open sets $\{U_n \times V_n\colon n \in E\}$ such that $x_n \in U_n$ and $x \in V_n$. It easy to see that $x = \bigcap\{V_n\colon n \in E\}$. □

A space X is called a quasi-$\varkappa$-space if every neighborhood of an arbitrary nonisolated point contains a countable nonclosed subset. Every $\varkappa$-space is a quasi-$\varkappa$-space. For example, all k-spaces, locally countably compact spaces, locally separable spaces, countably tight spaces are quasi-$\varkappa$-spaces.

Theorem 5.3. *If X is a regular quasi-$\varkappa$-space and is ∇-F_σ-sE, then X contains a dense subset of points of countable pseudocharacter.*

Corollary 5.4. *Every ∇-F_σ-sE dyadic space is metrizable.*

Theorem 5.3 is a consequence of the next Lemma 5.5, which proof is analogous to the proof of Lemma 3.2.

Lemma 5.5. *Let $\{y\} \cup M$ be a countable regular subspace of X with a nonisolated point y, and $Z = ((X \times M) \cup ((X \setminus \{y\}) \times \{y\})) \setminus \Delta$ is sE. Then y is a G_δ-point in X.*

Remark 5.6. It is impossible to omit the condition of being a $\varkappa$-space in Theorem 5.1 as the example $X = \beta\omega \setminus \omega$ shows. And it is also impossible to assert in Theorem 5.3 that the pseudocharacter of X is countable. The corresponding example is $X = \omega_1 + 1$.

6. Application to $\exp(X)$

The space of closed subsets $\exp(X)$ is the set of all nonempty closed subsets of X with the Vietoris topology [8, 2.7.20]. Applying Theorems 3.9 and 3.11 we shall prove

Theorem 6.1. *If $\exp(X)$ is a regular point-E space, then X is a hereditarily separable perfectly normal countably compact space.*

Lemma 6.2. *The space $\exp(\omega)$ is not point-E.*

Proof. The space $\exp(\omega)$ contains a closed homeomorph of the Sorgenfrey line S [6,19]. Hence $S \times S$ is a closed subset of $\exp(\omega)$ too. But it easy to see that $S \times S$ is not point-E. □

Lemma 6.3. *Let F and H be closed disjoint subsets of X such that H is infinite and $\exp(X)$ is regular and point-wE. Then F is a G_δ-point in $\exp(X)$.*

Proof. There exists an open set $U \supset F$, such that $\overline{U} \cap H = \emptyset$, because $\exp(X)$ is regular and hence X is normal. Then $\exp(H) \times \exp(\overline{U})$ is a closed subset of $\exp(X)$ and hence this product is point-wE. It is clear that $\omega + 1 \subset \exp(H)$. Applying Theorem 3.11 we have that F is a G_δ-point in $\exp(\overline{U})$ and consequently in $\exp(X)$. □

Lemma 6.4. *Let X be a countably compact space. If $\exp(X)$ is regular and point-wE, then a space X is hereditarily separable and perfectly normal.*

Proof. If X contains two nonisolated points $x \neq y$, one can choose open disjoint sets U and V such that $x \in U$ and $y \in V$. Then, if F is a closed subset of X, then $F = K \cup L$, where $K = F \setminus U$ and $L = F \setminus V$. The regularity of X and Lemma 6.3 imply that K and L are G_δ-points in $\exp(X)$ and hence they are G_δ-sets in X. Of course, it means that K and L are separable (Lemma 3 of [16]; see also [18]). Thus F is a separable G_δ-set in X. We see that the pseudocharacter of X is countable and X is countably compact, so X is first-countable. Hence every subset of X is separable [21, Proposition 3]. If X contains only one nonisolated point x, then X is the one-point compactification of the discrete space $X \setminus \{x\}$. It is easy to see that $X \times (\omega + 1) \subset \exp(X)$ in this case (see also Lemma 2 of [16]). Applying Theorem 3.11 we see that x is a G_δ-point, but it means that X is countable. □

Proof of Theorem 6.1. Lemma 6.2 implies countable compactness of X. Lemma 6.4 implies hereditary separability and perfect normality of X. □

Corollary 6.5. *If $\exp(\exp(X))$ or $\exp(X \times X)$ is regular and point-E, then X is a metrizable compact space.*

Corollary 6.6 [13]. *If $\exp(X)$ is hereditarily pseudonormal, then X is a hereditarily separable perfectly normal countably compact space.*

Corollary 6.7. *If* $\exp(X)$ *is* ∇*-normal, then* X *is a hereditarily separable perfectly normal compact space.*

Proof. Every ∇-normal space is point-normal and normal, so the space X is hereditarily separable and perfectly normal by Theorem 6.1 and is compact by Veličko's Theorem [8, 3.12.26(a), Remark]. □

Theorem 6.8. *Let* X *be a compact space. Then* $\exp(X)$ *is point-*wE *if and only if* X *is hereditarily separable and perfectly normal.*

Proof. If X is a compact space then $\exp(X)$ is a compact space too and is regular. Thus by Lemma 6.4 a space X is hereditarily separable and perfectly normal. If X is a hereditarily separable and perfectly normal compact space then $\exp(X)$ is first-countable [23] compact space. Thus $\exp(X)$ is point-(σ-compact) and hence is point-wE □

Problem 6.9. Is X a compact space, if $\exp(X)$ is regular point-E?

We note here that a space X is a metrizable compact space if $\exp(X)$ is hereditarily normal [4] or is regular hereditarily countably paracompact [15]. Thus the next problem seems to be natural.

Problem 6.10. Is X a metrizable compact space, if $\exp(X)$ is hereditarily pseudonormal?

Problem 6.11. Is X a metrizable compact space, if $\exp(X)$ is ∇-normal?

Problem 6.12. (a) Can we use point-wE in stead of point-E in Theorem 6.1? (b) Is it possible to weaken the regularity of $\exp(X)$ to the property of being a Hausdorff space in Theorem 6.1?

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